Suggested solution of HW4

Chapter 6 Q1: By the product formula of Γ, we have for all z,

$$\frac{1}{\Gamma(z)} = \Gamma(1-z)\frac{\sin \pi z}{\pi} = e^{\gamma z} z \prod_{n=1}^{\infty}(1+\frac{z}{n})e^{-z/n}$$

where γ is the Euler's constant.

$$\prod_{n=1}^{N} e^{\gamma z} z(1+\frac{z}{n})e^{-z/n} = e^{\gamma z} z \frac{(z+1)(z+2)\cdot\cdot(z+N)}{N!} e^{-z(\sum_{n=1}^{N}\frac{1}{n}-\log N)} \cdot e^{-z\log N}$$

$$\to \lim_{n\to\infty} \frac{z(z+1)\cdot\cdot(z+n)}{n^z n!},$$

whenever $z \neq 0, -1, -2, ...$

Chapter 6 Q3: Recall Walliss product formula stating that

$$\begin{aligned}
\frac{\pi}{2} &= \prod_{n=1}^{\infty}\left(\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\right) \\
&= \lim_{m\to\infty} \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5} ... \frac{2m\cdot 2m}{(2m-1)(2m+1)} \\
&= \lim_{m\to\infty} \frac{2^4\cdot 4^4 ... (2m)^4}{1^2\cdot 2^2\cdot 3^2\cdot ...(2m+1)^2}\cdot(2m+1) \\
&= \lim_{m\to\infty} \frac{(m!)^4 2^{4m}}{[(2m+1)!]^2}(2m+1).
\end{aligned}$$

Recall the Gauss formula,

$$\Gamma(s) = \lim_{n\to\infty}\frac{n^s n!}{s(s+1)...(s+n)}.$$

Direct substitution yields the following

$$\begin{aligned}
\frac{n^{2s+1/2}(n!)^2}{s(s+1/2)(s+1)...(s+n)(s+1/2+n)} &= \frac{4^{n+1}(n!)^2 n^{2s}\sqrt{n}}{(2s)(2s+1)...(2s+2n+1)} \\
&= \left[\frac{(2n+1)^{2s}(2n+1)!}{(2s)...(2s+2n+1)}\right]\cdot\frac{4^n\sqrt{2n+1}(n!)^2}{(2n+1)!}\cdot\frac{4n^{2s+1/2}}{(2n+1)^{2s+1/2}} \\
&\to \Gamma(2s)\cdot\sqrt{\frac{\pi}{2}}\cdot 2^{3/2-2s} = \sqrt{\pi}2^{1-2s}\Gamma(2s).
\end{aligned}$$

Chapter 6 Q4: For $f(z) = 1/(1-z)^{\alpha} = exp\,[-\alpha\log(1-z)]$, we have

$$(1-z)f'(z) = \ exp\,[-\alpha\log(1-z)] = \alpha f(z)$$

$$(1-z)f''(z) = (\alpha+1)f'(z)$$

Inductively, we have

$$(1-z)f^{(n)}(z) = (\alpha+n-1)f^{(n-1)}(z) = (\alpha+n-1)\cdot(\alpha+n-2)\cdot\cdot(\alpha+1)f(z).$$

Put $z = 0$, we have

$$f^{(n)}(0) = (\alpha + n - 1)f^{(n-1)}(z) = (\alpha + n - 1) \cdot (\alpha + n - 2) \cdot \cdot (\alpha + 1).$$

Since $a_n(\alpha) = f^{(n)}(0)/n!$, we have

$$\frac{a_n(\alpha)}{n^{\alpha-1}} = \frac{(\alpha + n - 1) \cdot (\alpha + n - 2) \cdot \cdot (\alpha + 1)}{n^{\alpha-1}} \to \Gamma(\alpha).$$

Chapter 6 Q10: (a) Our goal is to comparer the integral $\int_0^\infty e^{-t}t^{s-1}\, dt$ with $\int_0^\infty e^{-it}t^{s-1}\, dt$. So we consider the meromorphic function $f(w) = e^{-w}w^{z-1}$ for a fixed $z \in \mathbb{C}$ on $\{z : Re(z) \geq 0\}$.

Consider the contour suggested in textbook and denote it by $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$ where γ_1 is on the real axis, γ_2 is the large circular arc, γ_3 is on the imaginary axis and γ_4 is the smaller γ_4. As it is holomorphic, by cauchy theorem,

$$0 = \oint_\gamma f(w)\, dw = \oint_{\gamma_1} + \oint_{\gamma_2} + \oint_{\gamma_3} + \oint_{\gamma_4} f(w)\, dw.$$

We compute them separately.

$$\oint_{\gamma_1} f(w)\, dw = \int_\epsilon^R e^{-t}t^{z-1}\, dt,$$

$$\oint_{\gamma_2} f(w)\, dw = \int_0^{\pi/2} e^{-Re^{i\theta}}(e^{i\theta})^{z-1}R^z\, d\theta.$$

$$\oint_{\gamma_3} f(w)\, dw = -\int_\epsilon^R e^{-it}(it)^{z-1}i\, dt = -i^z \int_\epsilon^R e^{-it}t^{z-1}\, dt,$$

$$\oint_{\gamma_4} f(w)\, dw = -\int_0^{\pi/2} e^{-\epsilon e^{i\theta}}(e^{i\theta})^{z-1}\epsilon^z\, d\theta.$$

Letting $\epsilon \to 0$, we see that the fourth term goes to 0 if $Re(z) > 0$. Let $R \to \infty$, we see that the second term goes to 0 as illustrated below.

$$\left| \int_0^{\pi/2} e^{-Re^{i\theta}}(e^{i\theta})^{z-1}R^z\, d\theta \right| \leq R^{Re(z)} \cdot \int_0^{\pi/2} e^{-R\cos\theta}\, d\theta.$$

Let $t = \cos\theta$, we have

$$\int_0^{\pi/2} e^{-R\cos\theta}\, d\theta = \int_0^1 \frac{e^{-Rt}}{\sqrt{1-t^2}}\, dt \to 0.$$

The convergence can be seen by splitting it into two parts, $\int_{1/2}^1, \int_0^{1/2}$, or simply using Lebesgue's monotone convergence theorem. Thus, if $Re(z) \in (0, 1)$, we have

$$\left| \int_0^{\pi/2} e^{-Re^{i\theta}}(e^{i\theta})^{z-1}R^z\, d\theta \right| \to 0$$

as $R \to \infty$. So we yield the following result.

$$\int_0^\infty e^{-it}t^{z-1}\, dt = i^{-z} \int_0^\infty e^{-t}t^{z-1}\, dt = e^{-i\pi z/2}\Gamma(z).$$

Part (a) follows from taking real part and imaginary part of this expression.

(b) For $t \in (0,1)$, we have $\sin t \leq t$. So, whenever $Re(z) \in (-1,1)$,

$$\int_0^1 \left|\frac{\sin t}{t^{1-z}}\right| \, dt \leq \int_0^1 \frac{1}{t^{-Re(z)}} \, dt < \infty.$$

On the other hand,

$$\int_1^\infty \frac{\sin t}{t^{1-z}} \, dt = -\cos 1 + (z-1) \int_1^\infty \cos(t) t^{z-2} \, dt.$$

Thus, $M(\sin)(z)$ define a analytic function on $-1 < Re(z) < 1$. By analytic continuation, the second equality holds on the large strip. Put $z = 0$ and $z = -1/2$ to obtain the conclusion.

Chapter 6,Q14: (a) First part follows from the Fundamental theorem of calculus. And the second part follows from integrating $\log x$.

(b) By (a), and monotonicity of $\Gamma(x)$, for large n, we have

$$\log n + \log \Gamma(n) = \log \Gamma(n+1) \geq \int_n^{n+1} \log \Gamma(t) \, dt = n \log n - n + c \geq \log \Gamma(n).$$

So, we have the following inequality,

$$n \log n - n + c \geq \log \Gamma(n) \geq (n-1) \log n - n + c$$

$$1 - \frac{1}{\log n} + \frac{c}{n \log n} \geq \frac{\log \Gamma(n)}{n \log n} \geq \frac{n-1}{n} - \frac{1}{\log n} + \frac{c}{n \log n}.$$

Take $n \to \infty$ to obtain the asymptotic behaviour. Indeed, one can observe $\log \Gamma(n) \sim n \log n + O(n)$ from the first inequality.

chapter 6, Q15:

$$\Gamma(s) = \int_0^\infty e^{-t} t^{s-1} \, dt = n^s \int_0^\infty e^{-nt} t^{s-1} \, dt;$$

That is

$$n^{-s} \Gamma(s) = \int_0^\infty e^{-nt} t^{s-1} \, dt.$$

Summing over all n yield

$$\begin{aligned}
\zeta(s)\Gamma(s) &= \sum_{n=1}^\infty \int_0^\infty e^{-nt} t^{s-1} \, dt \\
&= \int_0^\infty t^{s-1} \sum_{n=1}^\infty e^{-nt} \, dt \\
&= \int_0^\infty \frac{t^{s-1}}{e^t - 1} \, dt.
\end{aligned}$$

Chapter 6,Q16: You may follow the step suggested in the tutorial class on 3/11 by proving the Riemann's Functional Equation. Or use the hints suggested in the textbook as stated in the following.

By Q15, we can write

$$\Gamma(s)\zeta(s) = \int_0^\infty \frac{t^{s-1}}{e^t - 1}\, dt = \int_0^1 \frac{t^{s-1}}{e^t - 1}\, dt + \int_1^\infty \frac{t^{s-1}}{e^t - 1}\, dt$$

As the function in the second integral is of exponential decay, it defines a entire function. Now we take a look on the first integral. By expanding the power series of $z/(e^z - 1)$ at $z = 0$, we have

$$\frac{1}{e^z - 1} = \sum_{m=0}^{\infty} \frac{B_m}{m!} z^{m-1}$$

where

$$\limsup_{m\to\infty} \left|\frac{B_m}{m!}\right|^{1/m} = \frac{1}{2\pi}. \tag{1}$$

by Cauchy Hadamard theorem and the fact that radius of convergence of $z/(e^z - 1)$ is 2π. Using this, we are able to represent the first integral in the following form.

$$\int_0^1 \frac{t^{s-1}}{e^t - 1}\, dt = \sum_{m=0}^{\infty} \frac{B_m}{m!(s+m-1)}.$$

By the control of (1), the series converges uniformly on any compact set away from $\{1, 0, -1, ...\}$. It gives a meromorphic function on $\mathbb{C}$ with simple pole at $\{1, 0, -1, ...\}$. But since $1/\Gamma(s)$ is a entire funtion with simple zero at $\{0, -1, -2, ...\}$. So

$$\frac{1}{\Gamma(s)} \int_0^1 \frac{t^{s-1}}{e^t - 1}\, dt$$

define a meromorphic function with a simple pole at $s = 1$.

Chapter 7,Q1: By summation by part formula, we can reformulate the Dirichlet series in the following way,

$$\sum_{n=1}^{N} \frac{a_n}{n^s} = \frac{A_N}{N^s} - \sum_{n=1}^{N-1} A_n \left[\frac{1}{(n+1)^s} - \frac{1}{n^s}\right].$$

Noted that in general, the mean value theorem does not hold for holomorphic function. So we argue in the following way.

For a fixed $s \in \mathbb{C}$, define a holomorphic function $f(z) = z^{-s}$ on the right half plane. Consider the straight line γ from $n+1$ to n.

$$\begin{aligned}\left|\frac{1}{(n+1)^s} - \frac{1}{n^s}\right| &= |f(n+1) - f(n)| \\ &= \left|\int_n^{n+1} f'(t)\, dt\right| \\ &\le \int_n^{n+1} |f'(t)|\, dt \le \sup\{|f'(t)| : t \in [n, n+1]\}.\end{aligned}$$

Within $[n, n+1]$, for $Re(s) > 0$,

$$|f'(x)| = \frac{|s|}{|x^{s+1}|} \leq \frac{|s|}{x^{Re(s)+1}} \leq \frac{|s|}{n^{Re(s)+1}}$$

Combine this to the series, we obtain

$$\sum_{n=1}^{N-1} |A_n| \left| \frac{1}{(n+1)^s} - \frac{1}{n^s} \right| \leq \sum_{n=1}^{N-1} \frac{M|s|}{n^{Re(s)+1}}.$$

The series is absolutely convergent. So $\sum_{n=1}^{\infty} \frac{a_n}{n^s}$ converge. And the convergence is uniform on any compact set due to the above estimate.

Chapter 7,Q2: (a) We claim that for two absolutely convergent series $\sum a_n$, $\sum b_n$,

$$\sum_{k=1}^{\infty} a_k \cdot \sum_{m=1}^{\infty} b_m = \sum_{n=1}^{\infty} \sum_{mk=n} a_k b_m.$$

Let $\epsilon > 0$, there exists N such that,

$$\sum_{k=N}^{\infty} |a_k| + \sum_{m=N}^{\infty} |b_m| < \epsilon.$$

On the other hand,

$$\left| \sum_{k=1}^{N} a_k \cdot \sum_{m=1}^{N} b_m - \sum_{n=1}^{N^2} \sum_{mk=n} a_k b_m \right| \leq \sum_{n=N+1}^{N^2} \sum_{mk=n; m>N} |a_k b_m| + \sum_{n=N+1}^{N^2} \sum_{mk=n; k>N} |a_k b_m|.$$

Due to the symmetry, it suffices to estimate one of terms in the right hand side.

$$\begin{aligned} \sum_{n=N+1}^{N^2} \sum_{mk=n; k>N} |a_k b_m| &\leq \sum_{k=N+1}^{N^2} |a_k| \cdot \sum_{m=1}^{N} |b_m| \\ &\leq \sum_{k=N}^{\infty} |a_k| \cdot \sum_{m=1}^{\infty} |b_m| < \epsilon \left(\sum_{m=1}^{\infty} |b_m| \right). \end{aligned}$$

(b) Since $(\zeta(s))^2 = \sum_{k=1}^{\infty} \frac{1}{k^s} \cdot \sum_{m=1}^{\infty} \frac{1}{m^s} = \sum_{k,m} \frac{1}{(mk)^s}$.

For each $n \in \mathbb{N}$, the number of divisors is equal to the number of ways of representing n by mk. Thus,

$$\sum_{k,m} \frac{1}{(mk)^s} = \sum_{n=1}^{\infty} \frac{d(n)}{n^s}.$$

Similarly,

$$\zeta(s)\zeta(s-a) = \sum_{k,m} \frac{m^a}{(mk)^s}$$

For a given $n \in \mathbb{N}$, we sum over all m^a where m is a divisor of n. Thus

$$\zeta(s)\zeta(s-a) = \sum_{n=1}^{\infty} \frac{\sigma_a(n)}{n^s}.$$

Chapter 7-Q3: (a) By Euler's identity,

$$\frac{1}{\zeta(s)} = \prod_p (1 - \frac{1}{p^s}) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}.$$

If $n = 1$, $a_1 = 1$ clearly. If $n = p_1 p_2 \cdot \cdot p_k$, where p_j are all distinct prime. Then $a_n = (-1)^k$. Result follows from unique factorization of integer.

(b) Since $1 = \zeta(s) \cdot \frac{1}{\zeta(s)}$,

$$1 = \sum_{m=1}^{\infty} \frac{1}{m^s} \cdot \sum_{k=1}^{\infty} \frac{\mu(k)}{k^s} = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

where $a_n = \sum_{mk=n} \mu(k) = \sum_{k|n} \mu(k)$. By comparing coefficient, we show the claim.

Chapter 7-Q5: (a) This is a special case of Q1 in Ch7.

(b)

$$\zeta(s) - \tilde{\zeta}(s) = \sum_{n=1}^{\infty} \frac{1}{(2n)^s} = \frac{1}{2^s} \sum_{n=1}^{\infty} \frac{1}{n^s} = 2^{-s}\zeta(s).$$

Rearrange it to obtain the desired equation.

(c) For $s \in (0, 1)$, since

$$\frac{1}{(2k-1)^s} - \frac{1}{(2k)^s} > 0$$

for all $k \in \mathbb{N}$, we have $\tilde{\zeta}(s) > 0$. By analytic continuation, equation in part (b) holds on the whole complex plane. So, $\zeta(s)$ has no zero on $(0, 1)$.
According to the functional equation,

$$\zeta(s) = \pi^{s-1/2}\zeta(1-s)\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}.$$

At $s = 0$, the simple pole at $s = 0$ for $\zeta(1 - s)$ is cancelled out with the simple zero of $\Gamma(s/2)$. Thus, ζ has no zero at $s = 0$.

Chapter 7-Q6: If $a = 1$,

$$\int_{c-iN}^{c+iN} \frac{1}{s}\, ds = \log\left[\frac{c+iN}{c-iN}\right] \to \log(-1) = i\pi.$$

If $a \in [0, 1)$, Since a^s/s has no pole on the right half plane of the vertical strip,

$$\frac{1}{2\pi i} \oint_{\gamma} \frac{a^s}{s} ds = 0,$$

where γ is the curve composed of the vertical strip from $c-iN$ to $c+iN$ and a semi-circle γ' from $c+iN$ to $c-iN$ in clockwise direction.
Thus,

$$\lim_{N\to\infty}\frac{1}{2\pi i}\int_{c-iN}^{c+iN}\frac{a^s}{s}\,ds=\lim_{N\to\infty}\frac{1}{2\pi i}\int_{-\gamma'}\frac{a^s}{s}\,ds.$$

But

$$\begin{aligned}\left|\int_{-\gamma'}\frac{a^s}{s}\,ds\right| &\le \int_{-\pi/2}^{\pi/2}\frac{|a^{c+N\cos t}|}{|c+Ne^{it}|}\cdot N dt\\ &\le \int_{-\pi/2}^{\pi/2}\frac{|a^{c+N\cos t}|}{N-c}\cdot N dt\\ &\le 4a^c\int_0^{\pi/2}a^{N\cos t}\,dt,\end{aligned}$$

for all large N. Via change of coordinate $\cos t=x$, we get

$$\int_0^1 a^{Nx}\frac{1}{\sin t}\,dx=\int_0^1\frac{a^{Nx}}{\sqrt{1-x^2}}\,dx\to 0\ \text{ as } N\to\infty.$$

You can show this by splitting the integral into $\int_{1/2}^1$ and $\int_0^{1/2}$. And followed by direct integration. Or use dominated convergence theorem. Combine everything to conclude the case of $a\in[0,1)$.

If $a>1$, consider the contour γ composed of the vertical strip from $c-iN$ to $c+iN$, the horizontal strip from $c+iN$ to y-axis, the semi-circle from iN to $-iN$ oriented in the anticlockwise direction, and then the horizontal strip from $-iN$ to the vertical strip $c+it, t\in\mathbb{R}$. Denote the contour γ by $\gamma_1+\gamma_2+\gamma_3+\gamma_4$ respectively.

For γ_3, we have

$$|\int_{\gamma_3}\frac{a^s}{s}\,ds|\le\int_{3\pi/2}^{\pi/2}\frac{a^{N\cos t}}{|Ne^{it}|}N\,dt=\int_{3\pi/2}^{\pi/2}a^{N\cos t}\,dt.$$

Argue similar to above, we have when $N\to\infty$

$$\int_{3\pi/2}^{\pi/2}a^{N\cos t}\,dt\to 0.$$

Similarly, the integral over γ_2,γ_4 also tends to 0, as $N\to\infty$. By residue theorem, the integral over γ is 1 which yield our result.

Chapter 7-Q8: (a)

$$F(-s)=\xi(1/2-s)=\xi(1-1/2+s)=\xi(1/2+s)=F(s).$$

The second equality follows from functional equation of ξ. By expanding its power series, $F(s)=\sum_{n=0}^{\infty}a_ns^n$. We conclude that $a_n=0$ if n is odd. So $G=\sum_{n=0}^{\infty}a_{2n}s^n$.

(b) The only pole of ζ is at $s = 1$, so it remains to show the order of growth of $(s-1)\zeta(s)$. Make use of the following functional equation, for $Re(s) < 0$,

$$\zeta(s) = \pi^{s-1/2}\zeta(1-s)\frac{\Gamma((1-s)/2)}{\Gamma(s/2)}.$$

Noted that $\frac{1}{\Gamma(s)}, \pi^s$ is of order 1.

$$\begin{aligned} |\Gamma((1-s)/2)| \leq \int_0^\infty e^{-t}|t^{(-s-1)/2}|\, dt &\leq \int_0^\infty e^{-t}t^{(-Re(s)-1)/2}\, dt \\ &\leq \Gamma((1-Re(s))/2) \\ &\leq (-Re(s))! \end{aligned}$$

By Stirling formula, it is of order 1. So it suffices to consider $Re(s) \geq 0$. But it has already been done in Proposition 2.7 of ch6.

It remains to show the lower bound. Substitute $s = -2n-1$, we have

$$\begin{aligned} |\zeta(-2n-1)| &= \pi^{-2n-3/2}\zeta(2+2n)\frac{\Gamma(n+1)}{\Gamma(-n-1/2)} \\ &= \pi^{-2n-3/2}\frac{n!}{\Gamma(-n-1/2)}\sum_{k=1}^{\infty}\frac{1}{k^{2+2n}}. \end{aligned}$$

On the other hand,

$$\begin{aligned} |\Gamma(-n-1/2)| &= \frac{\sqrt{\pi}}{(n+1/2)(n+1/2-1)\cdots(3/2)(1/2)} \\ &= \frac{2^n\sqrt{\pi}}{(2n+1)(2n-1)\cdots(3)(1)} \\ &= \frac{2^{2n}n!\sqrt{\pi}}{(2n)!}. \end{aligned}$$

Also, by comparing to the integral , we have

$$\sum_{k=1}^{\infty}\frac{1}{k^{2+2n}} \approx \frac{1}{1+2n}.$$

Combine all this, we get

$$|\zeta(-2n-1)| \approx \pi\frac{(2n)!}{(2\pi)^{2n}}\frac{1}{1+2n}$$

Taking log yield

$$\log|\zeta(-2n-1)| \approx \log\pi + \log[(2n)!] - 2n\log(2\pi) - \log(1+2n)$$

Since $\log(2n)! \approx 2n\log(2n) + O(n)$, we conclude that $\zeta(-2n-1)$ is of order 1.

(c) By Q14 in ch5, G has infinitely many zeros. Since

$$\zeta(s+1/2) = \pi^{1/4+s/2}\frac{\xi(s+1/2)}{\Gamma(s/2+1/4)} = \pi^{1/4+s/2}\frac{F(s)}{\Gamma(s/2+1/4)},$$

$\zeta(s)$ has infinitely many zeros.